

Majorana Fermion Codes

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Abstract.

We initiate the study of Majorana fermion codes. These codes can be viewed as extensions of Kitaev's 1D model of unpaired Majorana fermions in quantum wires to higher spatial dimensions and interacting fermions. The purpose of Majorana fermion codes (MFCs) is to protect quantum information against low-weight fermionic errors, that is, operators acting on sufficiently small subsets of fermionic modes. We examine to what extent MFCs can surpass qubit stabilizer codes in terms of their stability properties. A general construction of 2D MFCs is proposed which combines topological protection based on a macroscopic code distance with protection based on fermionic parity conservation. Finally, we use MFCs to show how to transform any qubit stabilizer code to a weakly self-dual CSS code.

1. Introduction

The physical realization of systems with topological quantum order and their theoretical description have been a topic of significant attention lately. This attention is partly motivated by the potential use of topologically ordered systems as fault-tolerant hardware in a quantum computer [1]. Encoding of quantum information into the ground states of such systems permits exponential suppression of dephasing, while the presence of a gap above the ground state suppresses thermal error excitations. The original insights [1, 2] concerning the zero-temperature stability of topologically ordered systems have been fully rigorously proved for quite general quantum spin systems in Ref. [3, 4].

A more ambitious goal for robust quantum information processing is to genuinely store and manipulate quantum information at nonzero temperature $T > 0$. Whether this is physically possible is the topic of the discussion on ‘self-correcting’ quantum memories [5, 6, 7], which do not need to be continuously error-corrected as the standard theory of quantum fault-tolerance prescribes [8]. The question of self-correction goes under alternative guises such as the question of thermal stability [9, 10] or thermal fragility [11] of quantum memories or the persistence of topological order at finite-temperature [12, 13, 14].

In understanding aspects of topological order or its possible extension to finite temperature, it is important to study physical ‘toy’ models such as the well-known surface code family or general quantum error-correcting codes with geometrically-local generators. These toy models both teach us what may be possible at the conceptual level as well as pose an interesting challenge to engineer these Hamiltonians at the physical level.

In this paper we introduce a class of toy models that can be viewed as extensions of Kitaev’s 1D model of unpaired Majorana fermions in quantum wires [15] to higher spatial dimensions and to interacting fermions. These toy models which we call *Majorana fermion codes* can be described by fermionic term-wise commuting Hamiltonians composed of geometrically-local interactions on a D -dimensional lattice. The purpose of Majorana fermion codes is to protect quantum information against low-weight fermionic errors, i.e., operators acting only on sufficiently small subsets of fermionic modes. One distinction between fermionic systems and the systems composed of qubits or spins is the presence of superselection rules. In particular, if a fermionic system interacts with a bosonic environment, conservation of the parity of the total number of fermions restricts the set of physically realizable errors to the so-called even fermionic operators.

One question addressed in the present paper is whether the superselection rules permit more robust storage of quantum information based on Majorana fermion codes as compared with qubit stabilizer codes under the same geometric locality constraints. We partially answer this question in the negative by generalizing the no-go theorem for quantum self-correction based on 2D stabilizer codes [7] to Majorana fermion codes.

On the positive side, we construct interesting 2D generalizations of Kitaev’s model of unpaired Majorana fermions in quantum wires [15]. Specifically, we construct a

family of Majorana fermion codes encoding one qubit into a 2D lattice of fermions folded into a cylinder. The corresponding Hamiltonian has exactly two zero-energy Majorana modes, i.e., odd fermionic operators \bar{C}_0, \bar{C}_1 supported on the opposite edges of the lattice, commuting with the Hamiltonian, and anti-commuting with each other. In coding theory language, \bar{C}_0 and \bar{C}_1 are the logical operators of the code. The main advantage of the 2D model is that the logical operators \bar{C}_0, \bar{C}_1 have a macroscopic weight proportional to the radius of the cylinder. It endows the encoded qubit with an extra degree of protection related to the macroscopic distance of the code and which does not rely on superselection rules. By varying the radius and the length of the cylinder one can combine the two types of protection in a controllable way.

The additional protection by the code distance is completely analogous to the protection of quantum information encoded into the ground state of topologically ordered systems as discussed in [1, 2, 3, 4]. Such protection might be necessary for example if one tries to build a multi-qubit register composed of 1D quantum wires with unpaired Majorana modes. In this case a perturbation can couple the unpaired modes that belong to adjacent wires without violating the superselection rules and hence an additional protection using a Majorana fermion code with a large distance could be helpful. In addition, the superselection rule prohibiting odd error operators is not likely to be completely rigorous, for instance, if the environment supports gapless fermionic modes that can couple to the system, or, when a single unpaired fermion could jump from the superconductor onto the topological insulator (although such processes are energetically suppressed by a gap at low-enough temperature).

Finally, we argue that Majorana fermion codes can be used as a tool to prove new facts about qubit stabilizer codes. In particular, we prove that any stabilizer code can be locally mapped onto a weakly self-dual Calderbank-Shor-Steane (CSS) code. The mapping preserves all parameters of the code such as the number of physical and logical qubits, the distance, and locality of the generators up to a constant factor.

An open problem which is not addressed in our paper is by what physical means and mechanisms (interacting) Majorana fermion codes can be realized. One expects that, similar as for spin-systems, such models could emerge as effective many-body Hamiltonians for interacting fermion systems which are treated with a perturbative or renormalization-group flow analysis. There is ample physical evidence that 2D strongly-correlated electron systems support topological order; the question is whether the Majorana fermion code framework can help in understanding how such topological order emerges from the basic physical interactions.

The paper is organized as follows. Section 2 reviews Kitaev's 1D model of unpaired Majorana fermions and provides some motivation behind the present work. Section 3 introduces notations and necessary facts from the theory of stabilizer codes. A formal definition of Majorana fermion codes is given in Section 4. The mappings between stabilizer codes and Majorana fermion codes are described in Section 5. This section also proves the equivalence between general stabilizer codes and weakly self-dual CSS codes. In Section 6 we discuss quantum error correction in the presence of superselection

rules. The generalization of Kitaev's 1D model with unpaired Majorana fermions is described in Section 7. We give more examples of Majorana fermion codes that possess odd logical operators in Section 8.

2. Why Majorana fermion codes?

In Ref. [15] Kitaev considered the toy Hamiltonian of a 1D chain of spinless fermions interacting with a superconductor. The interaction with the superconductor allows for the creation and annihilation of pairs of fermions so that the total Hamiltonian of the system preserves only the parity of the number of fermions. Any Hamiltonian involving spin or spinless fermions can be written in terms of Majorana fermion operators by taking $a_k = \frac{1}{2}(c_{2k-1} + ic_{2k})$ and $a_k^\dagger = \frac{1}{2}(c_{2k-1} - ic_{2k})$. The Hermitian Majorana operators obey the relations

$$c_i c_j + c_j c_i = 2\delta_{ij} I.$$

For a particular choice of couplings, Kitaev's Hamiltonian on L spinless fermions, hence $2L$ Majorana fermions, reads

$$H = i \sum_{j=1}^{L-1} c_{2j} c_{2j+1}, \quad (1)$$

i.e. all Majorana modes are 'paired', except the first mode c_1 and the last c_{2L} . Thus this quadratic fermion Hamiltonian commutes with the operators c_1 and c_{2L} which we identify as the logical operators of a protected qubit. Alternatively, it can be said that the symmetry gives rise to the presence of a pair of zero-energy Majorana boundary modes which lead to degeneracy at the Fermi-level. Recently, there have been various proposals to realize such 1D model with zero-energy Majorana fermions, for example at the boundary between a topological insulator and a superconductor [16] or in a semiconducting heterostructure [17].

Ground states of H are -1 eigenvectors for every term $ic_{2j}c_{2j+1}$ in Eq. (1). The two-fold degenerate ground subspace has a basis $|\bar{0}\rangle$ and $|\bar{1}\rangle$ which satisfies

$$ic_1 c_{2L} |\bar{0}\rangle = |\bar{0}\rangle \quad \text{and} \quad ic_1 c_{2L} |\bar{1}\rangle = -|\bar{1}\rangle.$$

The logical Pauli operators for a qubit encoded into $|\bar{0}\rangle$ and $|\bar{1}\rangle$ can be chosen as $\bar{X} = c_1$, $\bar{Y} = -c_{2L}$, and $\bar{Z} = ic_1 c_{2L}$. Note that two of the logical operators c_1 and c_{2L} are of *odd* weight and hence would require the coherent creation/annihilation of a single fermion which is prohibited by superselection. An essential part of the model is that the only even-weight logical operator $c_1 c_{2L}$ is very non-local. It is natural to assume that elementary perturbations to the Hamiltonian and errors can be represented by local even weight Majorana fermion operators. Hence in a perturbative analysis such as the Schrieffer-Wolf perturbation theory, the first contributions that split the energy degeneracy between $|\bar{0}\rangle$ and $|\bar{1}\rangle$ are expected to occur in $O(L)$ th order implying that the splitting in degeneracy between $|\bar{0}\rangle$ and $|\bar{1}\rangle$ is exponentially small in L .

The spectrum and properties of Kitaev's model –as any other quadratic fermion Hamiltonian in the theory of topological insulators– are efficiently computable and quantum circuits which employ only non-interacting fermion Hamiltonians and simple fermionic measurements are efficiently simulatable classically, see [18]. Hence, if we are serious about using fermionic systems to robustly store and manipulate quantum information (see e.g. [19]), we will need some source of interaction to obtain quantum universality (see e.g. [20]). Let us mention that generalizations of the Hamiltonian Eq. (1) to interacting fermions have been recently considered by Fidkowski and Kitaev [21] to study the effect of interactions on the classification of 1D topological insulators.

The toy model, Eq. (1), demonstrates that fermionic parity conservation provides an alternative protection mechanism for the encoded qubit unrelated to topological quantum order. It is therefore natural to ask whether topological protection can be combined with protection based on superselection rules in an advantageous way. The Majorana fermion codes introduced in the present paper provide a natural framework in which such a hybrid protection can be studied.

3. Definitions and notations

We review a few standard definitions and notations which are used in this paper. Let X_i, Y_i, Z_i represent the three Pauli matrices on qubit i . Let $\mathcal{P}_n = \langle iI, X_1, Z_1, \dots, X_n, Z_n \rangle$ be the Pauli group on n qubits generated by single-qubit Pauli operators and the phase factors $\pm 1, \pm i$. The support of a Pauli operator P , $\text{Supp}(P)$ is the set of qubits on which it acts non-trivially. The size of the support, $|\text{Supp}(P)|$, is also sometimes called the weight of P , denoted as $|P|$.

A pair of logical operators for an encoded qubit is denoted as (\bar{X}, \bar{Z}) and $\bar{Y} = i\bar{X}\bar{Z}$. A stabilizer code is determined by its *stabilizer group* $\mathcal{S} \subseteq \mathcal{P}_n$ which is an Abelian subgroup of \mathcal{P}_n . To preclude \mathcal{S} from containing non-trivial phase factors one usually adds a requirement $-I \notin \mathcal{S}$. The set of Pauli operators $P \in \mathcal{P}_n$ that commute with all elements of \mathcal{S} is called the *centralizer* of \mathcal{S} and is denoted as $\mathcal{C}(\mathcal{S})$. If the stabilizer group is generated by $n - k$ independent generators, then the centralizer is generated by $n + k$ independent generators. Logical operators of a stabilizer code \mathcal{S} are elements of $\mathcal{C}(\mathcal{S})$ which are not in \mathcal{S} . One can always choose a set of $2k$ logical Pauli operators $\bar{X}_1, \bar{Z}_1, \dots, \bar{X}_k, \bar{Z}_k \in \mathcal{C}(\mathcal{S}) \setminus \mathcal{S}$ obeying the usual Pauli commutation relations. Note that $\mathcal{C}(\mathcal{S}) = \langle i, \mathcal{S}, \bar{X}_1, \bar{Z}_1, \dots, \bar{X}_k, \bar{Z}_k \rangle$. Given a stabilizer code \mathcal{S} , its codespace is spanned by all n -qubit states invariant under the action of \mathcal{S} . In this case the codespace is isomorphic to a space of k qubits, the so called *logical qubits*.

The distance d of a stabilizer code is defined as the minimum weight of a logical operator, i.e.

$$d = \min_{P \in \mathcal{C}(\mathcal{S}) \setminus \mathcal{S}} |P|. \quad (2)$$

A stabilizer code which encodes k logical qubits into n physical qubits and has distance

d is denoted as a $[[n, k, d]]$ code. We shall be interested in geometrically-local stabilizer codes and similarly local Majorana fermion codes. For such codes, qubits occupy sites of a D -dimensional lattice (or, more generally, some graph equipped with a metric) and the stabilizer group has a set of geometrically-local generators, $\mathcal{S} = \langle S_1, \dots, S_m \rangle$, that is, the support of any generator S_i has diameter at most $r = O(1)$ with respect to the lattice geometry \ddagger .

With a geometrically-local stabilizer code \mathcal{S} we can associate a Hamiltonian, e.g. $H_{\mathcal{S}} = -\sum_i S_i$ where S_i is an (over)complete set of stabilizer generators for \mathcal{S} . Then the ground-space of the Hamiltonian corresponds to the codespace and one may consider properties of such physical system at zero or non-zero temperature T .

Calderbank-Shor-Steane (CSS) codes are a particular subclass of stabilizer codes for which the stabilizer group \mathcal{S} can be represented as a product of two subgroups, $\mathcal{S} = \mathcal{S}(X) \cdot \mathcal{S}(Z)$, that contain only X -type and Z -type Pauli operators respectively. Any CSS code can be specified by a pair of classical linear codes $C_X, C_Z \subseteq \{0, 1\}^n$ such that

$$\mathcal{S}(X) = \{P = \prod_{i=1}^n X_i^{x_i} : (x_1, \dots, x_n) \in C_X\},$$

and

$$\mathcal{S}(Z) = \{P = \prod_{i=1}^n Z_i^{z_i} : (z_1, \dots, z_n) \in C_Z\}.$$

Commutativity between $\mathcal{S}(X)$ and $\mathcal{S}(Z)$ is equivalent to the mutual orthogonality of the codes C_X, C_Z , that is, one must have $\sum_{i=1}^n x_i z_i = 0 \pmod{2}$ for all $x \in C_X$ and $z \in C_Z$. The special subclass of CSS codes obeying $C_X = C_Z$ are called weakly self-dual CSS codes. For such codes the stabilizer group is invariant under the exchange of X and Z operators on every qubit. In that case the code $C = C_X = C_Z$ must be a weakly self-dual classical code, that is, $\sum_{i=1}^n x_i z_i = 0 \pmod{2}$ for all $x, z \in C$. For more background on CSS codes, see [22].

4. Definition of Majorana fermion codes

We define a Majorana fermion code as follows. Let c_1, c_2, \dots, c_{2n} be the $2n$ Majorana modes, i.e., operators obeying commutation rules

$$c_u c_v + c_v c_u = 2\delta_{u,v} I, \quad c_u^\dagger = c_u.$$

The total number of Majorana modes ($2n$) is always even, because these modes are obtained from n original fermionic modes. The single-mode operators c_1, \dots, c_{2n} , together with the phase factor i , generate a group of Majorana operators $\text{Maj}(2n)$.

\ddagger Here and below the notation $f = O(g)$ refers to the limit $n \rightarrow \infty$. In particular, $r = O(1)$ means that r is upper bounded by a constant independent of n .

Any element of the group $\text{Maj}(2n)$ can be represented as ηc_A where $A \subseteq \{1, \dots, 2n\}$ is some subset of modes,

$$c_A = \prod_{u \in A} c_u, \quad (3)$$

and $\eta \in \{\pm 1, \pm i\}$ is a phase factor. The subset A is called the support of c_A . Here and below we use a standard ordering of the product of single-mode operators c_u , meaning that the indices u increase from the left to the right. We define the weight of a Majorana operator as the number of modes in its support, that is, $|c_A| = |A|$. A Majorana operator is called even (odd) iff its support has even (odd) size.

For two arbitrary supports A and B we have

$$c_A c_B = (-1)^{|A| \cdot |B| + |A \cap B|} c_B c_A. \quad (4)$$

When either c_A or c_B is even, the commutation relation only depends on the parity of overlap $|A \cap B|$. In particular, when regions A and B do not overlap, i.e. $|A \cap B| = 0$, then Majorana operators commute, as is trivially the case for Pauli operators on non-overlapping supports. However, when c_A and c_B are both odd and their supports are non-overlapping, then c_A and c_B anti-commute.

One can always map the Majorana modes onto Pauli operators on n qubits using the Jordan-Wigner transformation $\Upsilon : \text{Maj}(2n) \rightarrow \mathcal{P}_n$ defined as $\Upsilon(c_{2i-1}) = Z_1 Z_2 \dots Z_{i-1} X_i$ and $\Upsilon(c_{2i}) = Z_1 \dots Z_{i-1} Y_i$, $i = 1, \dots, n$. Accordingly, the Hilbert space \mathcal{F}_n describing $2n$ Majorana modes can be identified with the Hilbert space of n qubits. For $D > 1$ systems, the Jordan-Wigner transformation generically maps local Majorana operators onto non-local Pauli operators. This is one of the reasons that local Majorana fermion codes defined below may exhibit different properties than local stabilizer codes (see e.g. [20, 23] for means to make this mapping local by introducing additional modes).

A Majorana fermion code is determined by its stabilizer group $\mathcal{S}_{\text{maj}} \subseteq \text{Maj}(2n)$ which must obey two conditions:

- \mathcal{S}_{maj} is an Abelian group not containing $-I$.
- All elements of \mathcal{S}_{maj} have even weight.

The second condition guarantees that stabilizer operators preserve the parity of the number of fermions in the system, and thus any element of \mathcal{S}_{maj} is a physically realizable operation. Given any pair of stabilizer operators proportional to c_A and c_B , the commutativity $c_A c_B = c_B c_A$ and the even-weight condition imply that the overlap $|A \cap B|$ must be even, see Eq. (4).

The set of Majorana operators $P \in \text{Maj}(2n)$ that commute with all elements of \mathcal{S}_{maj} is called the *centralizer* of \mathcal{S}_{maj} and is denoted as $\mathcal{C}(\mathcal{S}_{\text{maj}})$. Note that the centralizer may contain both even and odd Majorana operators. Logical operators of a Majorana fermion code \mathcal{S}_{maj} are elements of $\mathcal{C}(\mathcal{S}_{\text{maj}})$ which are not in \mathcal{S}_{maj} . If \mathcal{S}_{maj} is generated by $n - k$ independent generators, then $\mathcal{C}(\mathcal{S}_{\text{maj}})$ is generated by $n + k$ independent generators. One can always choose a set of $2k$ logical Pauli operators $\bar{X}_1, \bar{Z}_1, \dots, \bar{X}_k, \bar{Z}_k \in \mathcal{C}(\mathcal{S}_{\text{maj}}) \setminus \mathcal{S}_{\text{maj}}$ obeying the usual Pauli commutation relations. Note

that $\mathcal{C}(\mathcal{S}_{\text{maj}}) = \langle i, \mathcal{S}_{\text{maj}}, \bar{X}_1, \bar{Z}_1, \dots, \bar{X}_k, \bar{Z}_k \rangle$. In general the set of logical operators may contain both even and odd Majorana operators. We give necessary and sufficient conditions for a code to contain odd logical operators in Section 5, see Proposition 1. The codespace of a Majorana fermion code \mathcal{S}_{maj} is the linear subspace of \mathcal{F}_n spanned by all states invariant under the action of \mathcal{S}_{maj} .

From now on, we shall be interested in geometrically-local Majorana fermion codes. For such codes Majorana modes c_u occupy sites of some D -dimensional lattice Λ and the stabilizer group \mathcal{S}_{maj} has a set of geometrically-local generators, $\mathcal{S}_{\text{maj}} = \langle S_1, \dots, S_m \rangle$, that is, the support of any generator S_i has diameter at most $r = O(1)$. The codespace of \mathcal{S}_{maj} coincides with the ground subspace of a Hamiltonian $H = -\sum_i S_i$ that involves only geometrically-local interactions among the Majorana modes.

The simplest example of a local Majorana fermion code is the one describing Kitaev's 1D model of Eq. (1). The corresponding stabilizer group is

$$\mathcal{S}_{\text{maj}} = \langle ic_2c_3, \dots, ic_{2n-2}c_{2n-1} \rangle, \quad (5)$$

while the logical operators of the code can be chosen as $\bar{X} = c_1$ and $\bar{Z} = c_{2n}$. We describe a 2D generalization of this code in Section 7 and give some other examples of local Majorana fermion codes in Section 8.

We can define the distance of a Majorana fermion code similar to the distance of stabilizer codes, i.e. as the minimum weight of logical operators,

$$d = \min_{C \in \mathcal{C}(\mathcal{S}_{\text{maj}}) \setminus \mathcal{S}_{\text{maj}}} |C|. \quad (6)$$

According to this definition, a code with distance d is able to detect any error affecting less than d Majorana modes, i.e., any operator c_A with $|A| < d$ is a detectable error.

The notion of a code's distance does not completely capture all aspects of stability that Majorana fermion codes can offer since it treats even and odd logical operators on the same footing. However, if the system is closed or interacts with a bosonic environment, all physically realizable perturbations and error operators must preserve fermionic parity and thus must be even. In order to measure the degree of protection based on the superselection rules, let us introduce an additional parameter l_{even} defined as the minimum diameter of a region that can support an even logical operator,

$$l_{\text{even}} = \min_{\substack{C \in \mathcal{C}(\mathcal{S}_{\text{maj}}) \setminus \mathcal{S}_{\text{maj}} \\ |C| \equiv 0 \pmod{2}}} \text{diam}(\text{Supp}(C)). \quad (7)$$

As far as zero-temperature stability is concerned, the parameter l_{even} determines the smallest order of perturbation theory at which the ground state degeneracy of the code Hamiltonian $H = -\sum_i S_i$ can be lifted by a perturbation that involves only even geometrically local operators. Hence the parameters d and l_{even} capture two independent mechanisms of protection: topological protection by the code distance and protection based on the superselection rules respectively.

Consider as an example the code defined in Eq. (5). Obviously, it has distance $d = 1$. Meanwhile, any even logical operator must include both c_1, c_{2n} , and therefore

$l_{\text{even}} = 2n$ (the lattice size). More generally, using Lemma 2 from Section 5 and the fact (see [7]) that any 1D stabilizer code has distance $O(1)$, one can easily show that the distance of any 1D Majorana fermion code is $O(1)$.

It is important to note that the minimum weight logical operator C in Eq. (6) always has a connected support, that is, one cannot decompose C as $C = c_A c_B$ where the separation between A and B is larger than r (the largest diameter of the generators of \mathcal{S}_{maj}). Indeed, in this case both c_A and c_B would individually commute with \mathcal{S}_{maj} . Hence c_A or c_B would be a logical operator which contradicts the minimality of C . On the other hand, the minimum weight even logical operator may have highly disconnected support as the code in Eq. (5) demonstrates.

A simple argument shows that any Majorana fermion code has even logical operators. Indeed, if $\bar{X}, \bar{Y}, \bar{Z}$ are logical Pauli operators for some encoded qubit then the identity $\bar{X}\bar{Y}\bar{Z} \propto I$ implies that either all $\bar{X}, \bar{Y}, \bar{Z}$ are even, or two of them are odd and the third one is even. We describe properties of even logical operators for general D -dimensional Majorana fermion codes in Section 6.

As mentioned earlier, the superselection rule prohibiting odd error operators is not likely to be completely rigorous. For instance, one might be interested in constructions of Majorana fermion codes in which both d and l_{even} can be made arbitrarily large by increasing lattice dimensions, see Section 7.8.

Before we continue, let us make a few remarks about general Majorana fermion codes based on *non-interacting* fermions. We would like to make the point that going to higher spatial dimensions $D > 1$ does, in one aspect, not lead to fundamentally different behavior as compared to the 1D Kitaev's model of unpaired Majorana fermions [15]. More specifically, the Bogoliubov transformation allows one to transform any non-interacting Majorana fermion Hamiltonian $H = i \sum_{k \neq l} \alpha_{kl} c_k c_l$ into a canonical form in which some subset of Majorana modes is unpaired (i.e. these modes do not enter into the Hamiltonian). Since each unpaired mode is a linear combination of the original Majorana operators c_k , the ground subspace of H can be regarded as a quantum code with distance $d = 1$. Hence non-interacting models can only offer protection based on superselection rules similar to what the 1D model of Ref. [15] achieves. On the other hand, unlike in 1D, in 2D non-interacting fermion systems with unpaired Majorana modes, one can imagine adiabatically changing the Hamiltonian (or 'deforming' the quantum code [24]) to move localized unpaired Majorana modes around and enact some (but not all) logical gates by braiding.

5. Code mappings

In this Section we describe inter-conversions between three classes of codes: (i) qubit stabilizer codes, (ii) Majorana fermion codes and (iii) weakly self-dual CSS codes.

Lemma 1 (Kitaev [25]) *Every $[[n, k, d]]$ qubit stabilizer code \mathcal{S} can be mapped onto a Majorana fermion code \mathcal{S}_{maj} on $4n$ modes encoding k logical qubits with distance $2d$.*

For completeness, we give the mapping:

Proof: With every qubit j , we associate four Majorana fermion modes, $b_j^{x,y,z}$ and c_j and hence we have a total of $4n$ Majorana fermions. In the $n - k$ independent stabilizer generators of \mathcal{S} , we replace the local Pauli operators by $X_j = ib_j^x c_j$, $Y_j = ib_j^y c_j$, $Z_j = ib_j^z c_j$. In addition, for each qubit j we add a stabilizer $D_j = b_j^x b_j^y b_j^z c_j$ to \mathcal{S}_{maj} (on the subspace for which $D_j = +1$ we have $X_j Y_j Z_j = iI$). Thus the Majorana fermion code \mathcal{S}_{maj} is generated by $2n - k$ independent generators, and therefore it encodes k logical qubits. Each logical operator of the stabilizer code corresponds to a logical operator of the Majorana code. Also, since a logical operator of the Majorana fermion code has to commute with each D_j , it must contain an even number of the set $\{b_j^x, b_j^y, b_j^z, c_j\}$, and therefore it corresponds to a logical operator of \mathcal{S} . Since every Pauli operator corresponds to a weight-2 Majorana operator, the distance of the Majorana fermion code is twice the distance of the stabilizer code. \square .

Note that by this mapping every operator in the stabilizer \mathcal{S}_{maj} and logical operator in $\mathcal{C}(\mathcal{S}_{\text{maj}})$ will have *even* weight. In addition, qubit errors get mapped onto even Majorana operators.

Lemma 2 (Doubling) *With every Majorana fermion code \mathcal{S}_{maj} on $2n$ Majorana modes which encodes k logical qubits and has distance d , we can associate a $[[2n, 2k, d]]$ weakly self-dual CSS code.*

Let us first illustrate the idea of the doubling map using the simplest Majorana code with 4 Majorana modes and a single generator $\mathcal{S}_{\text{maj}} = \langle c_1 c_2 c_3 c_4 \rangle$. Clearly this code has $k = 1$ logical qubit with logical operators $\bar{X}_1 = ic_1 c_2$ and $\bar{Z}_1 = ic_1 c_3$. One can easily check that \mathcal{S}_{maj} has distance $d = 2$. The doubled version of \mathcal{S}_{maj} is a $[[4, 2, 2]]$ stabilizer code with a stabilizer group $\mathcal{S} = \langle X_1 X_2 X_3 X_4, Z_1 Z_2 Z_3 Z_4 \rangle$ obtained by replacing each operator c_u either with X_u or Z_u (hence the number of generators is doubled). The logical operators of \mathcal{S} can be chosen as $\bar{X}_1 = X_1 X_2$, $\bar{Z}_1 = Z_1 Z_3$, $\bar{X}_2 = X_1 X_3$, and $\bar{Z}_2 = Z_1 Z_2$.

Proof of Lemma 2: Any operator $P \in \text{Maj}(2n)$ can be parameterized (up to a phase factor) by a binary string $x \in \{0, 1\}^{2n}$ such that multiplication in $\text{Maj}(2n)$ corresponds to addition of binary strings modulo two. Specifically, one sets $x_u = 1$ if u belongs to the support of P and $x_u = 0$ otherwise. Let $\phi : \text{Maj}(2n) \rightarrow \{0, 1\}^{2n}$ be the corresponding mapping. Consider a classical code

$$C = \phi(\mathcal{S}_{\text{maj}}) \subset \{0, 1\}^{2n}.$$

Note that $\dim(C) = n - k$, since \mathcal{S}_{maj} has $n - k$ independent generators. Furthermore, since \mathcal{S}_{maj} is an Abelian group containing only even operators, the supports of any elements $P, Q \in \mathcal{S}_{\text{maj}}$ must have even overlap. Hence C is a weakly self-dual classical code, that is, $\sum_{i=1}^{2n} x_i y_i = 0 \pmod{2}$ for any $x, y \in C$, or, equivalently,

$$C \subseteq C^\perp.$$

Let $\mathcal{S} = \mathcal{S}(X) \cdot \mathcal{S}(Z)$ be the weakly self-dual CSS code constructed from C as explained in Section 3. By construction, the code \mathcal{S} has $2n$ qubits, $n - k$ independent generators

of X -type, and $n - k$ independent generators of Z -type. Hence \mathcal{S} encodes $2k$ qubits. Note that each generator of \mathcal{S}_{maj} gives rise to a pair of generators in \mathcal{S} : the one obtained by replacing each single-mode operator c_j with X_j , and the one obtained by replacing each single-mode operator c_j with Z_j .

Consider a minimum-weight logical operator in the code \mathcal{S} ; w.l.o.g. it is either a product of X or a product of Z s (but not of both). When we replace each Pauli X_i (or Z_i) by a Majorana operator c_i , we obtain a logical operator for the Majorana code. Vice versa, every logical Majorana operator gives rise to a pair of logical operators for the stabilizer code \mathcal{S} . Hence the distances of these codes are identical. \square .

Combining the two lemmas, we get the following useful fact.

Corollary 1 *Any $[[n, k, d]]$ stabilizer code can be mapped onto a $[[4n, 2k, 2d]]$ weakly self-dual CSS code. This mapping preserves geometric locality of a code up to a constant factor.*

This result thus shows that in order to derive distance bounds for, say, geometrically-local codes, one only needs to prove such bounds for weakly self-dual CSS codes and the scaling of rates and relative overhead can be determined by considering only weakly self-dual CSS codes. In addition, the code mappings allow one to show that the partition function of a Hamiltonian associated with a stabilizer code can be expressed as the partition function of a classical Ising (\mathbf{Z}_2) gauge theory. Since the mapping preserves the locality of errors, it is also the physics which is preserved. Hence the properties of \mathbf{Z}_2 -gauge models, the presence of a phase-transition or not, will be related to the question of thermal stability of any stabilizer code [26]. The weakly self-dual character of the CSS code, or the fact that the Hamiltonian of the \mathbf{Z}_2 -gauge model has only terms with even overlap, is crucial. For example, it is well-known that there exists a 3D Ising gauge model [27] (in fact, this model is the Z -part (i.e. the subgroup $\mathcal{S}(Z)$ of the stabilizer \mathcal{S}) of the 3D surface code which was shown to be thermally stable against X -errors [28]) which has a phase-transition at a non-zero temperature T_c . However, this Ising gauge model does not have the property that all terms have even overlap. Hence this model is not directly pertinent to the thermal stability of 3D stabilizer codes for which one needs macroscopic energy barriers against both against X - and Z -error excitations.

6. Properties of Majorana fermion codes

In the previous section we have seen that any local Majorana fermion code can be mapped to a local weakly self-dual CSS code without changing parameters of the code in any significant way, see Lemma 2. Therefore one directly apply any distance upper bounds obtained for local stabilizer codes in [7, 29] to obtain analogous upper bounds on local Majorana fermion codes. However, one might expect that a Majorana fermion code may offer an additional degree of protection resulting from conservation of the fermionic parity. Such additional protection may only manifest itself for Majorana fermion codes that possess odd logical operators, since for such codes at least some subset of the

logical operators is protected by the superselection rules, see Section 4. Hence, the first question we address in this section is under what conditions a Majorana fermion code has at least one odd logical operator. Let us define an operator C_{all} measuring the parity of the total number of fermions,

$$C_{\text{all}} = i^n c_1 c_2 \cdots c_{2n-1} c_{2n}. \quad (8)$$

Proposition 1 *A Majorana fermion code \mathcal{S}_{maj} has at least one odd logical operator iff $C_{\text{all}} \notin \pm \mathcal{S}_{\text{maj}}$.*

Proof: Indeed, suppose $C_{\text{all}} \in \pm \mathcal{S}_{\text{maj}}$. Then any logical operator $P \in \mathcal{C}(\mathcal{S}_{\text{maj}}) \setminus \mathcal{S}_{\text{maj}}$ must have even weight, since it has to commute with C_{all} . Suppose now that $C_{\text{all}} \notin \pm \mathcal{S}_{\text{maj}}$. Since the support of C_{all} has even overlap with the support of any element of \mathcal{S}_{maj} , we conclude that C_{all} is a logical operator. Then there must exist another logical operator P which anti-commutes with C_{all} . But this is possible only if P has odd weight. \square

Let us point out that for any Majorana fermion code there is a choice of logical Pauli operators such that at most one of them is odd. Indeed, if a code has at least one odd logical operator, we can choose logical Pauli operators on the first encoded qubit as $\bar{Z}_1 = C_{\text{all}}$ and $\bar{X}_1 = P$, where P is some odd logical operator, see Proposition 1. Since the logical Pauli operators \bar{X}_i, \bar{Z}_i on the remaining qubit must commute with \bar{Z}_1 , they must be even. It will be convenient to introduce a parameter $k_{\text{odd}} \in \{0, 1\}$ such that $k_{\text{odd}} = 1$ iff a code has at least one odd logical operator. (It is perhaps important to note that $k_{\text{odd}} = 1$ does not imply that at most one logical qubit has odd logical operators. In fact, one can show that there always exists a choice for the logical operators such that *all* logical qubits have logical \bar{X} and \bar{Z} operators which are odd weight.)

For 2D stabilizer codes, it has been proved in [7] that one can always find logical operators which are supported on a strip of constant width, i.e., have string-like geometry. Lemma 2 immediately shows that the same result holds for 2D Majorana fermion codes. In particular, the distance of any 2D Majorana fermion code defined on a lattice of size $L \times L$ is at most $O(L)$. However, as we mentioned earlier in the paper, for Majorana fermion codes with $k_{\text{odd}} = 1$ we have to focus only on the even logical operators while odd logical operators are prohibited by the superselection rules. For example, the results of [7] do not rule out the possibility that a 2D Majorana fermion code may have logical operators $\bar{X}, \bar{Y}, \bar{Z}$ such that \bar{X} is even, \bar{Y}, \bar{Z} are odd, and \bar{X} has a plane-like geometry, that is, the minimum weight of \bar{X} is of order $n \sim L^2$. Such a code might behave similar to the classical 2D ferromagnetic Ising model in terms of its thermal stability. Unfortunately, below we prove that 2D Majorana fermion codes do not behave like the 2D ferromagnetic Ising model; more precisely we will show the following.

Lemma 3 *Let $\mathcal{S}_{\text{maj}} = \langle S_1, \dots, S_m \rangle$ be a local Majorana fermion code defined on a 2D lattice (with periodic or open boundary conditions) such that the support of any generator S_a has diameter at most $r - 1$ for some constant r . Let $\Lambda = A_1 \cup A_2 \cup \dots \cup A_t$ be a*

partition of the lattice into parallel disjoint strips of width at least r . Then one of the following (or both) is true:

- (i) There exists an even logical operator \bar{C} supported on some strip A_i ;
- (ii) There exists a pair of odd logical operators \bar{C}_i, \bar{C}_j supported on some pair of strips A_i, A_j , $i \neq j$.

Let us first comment on the implications of the lemma. Consider first the case (i). In this case the code has an even logical operator \bar{C} whose support is confined to a rectangular region of size $r \times L$, where L is the lattice size. In other words, at least one even logical operator has string-like geometry. Consider now the case (ii). Since the logical operators \bar{C}_i, \bar{C}_j are odd and have non-overlapping supports, they must anti-commute. It implies that \bar{C}_i and \bar{C}_j are distinct logical operators, that is, $\bar{C}_i \bar{C}_j \notin \mathcal{S}_{\text{maj}}$. But then $\bar{C}_i \bar{C}_j$ is an even logical operator whose support consists of two disjoint string-like regions. This result suggests that 2D Majorana fermion codes cannot surpass 2D stabilizer codes in terms of their thermal stability by gaining additional protection based on the superselection rules. Indeed, as was pointed out by many authors [5, 6, 30, 7, 9], the existence of string-like logical operators and the lack of string-tension rule out the possibility of quantum self-correction at a non-zero temperature. More in particular, a logical operator supported on two string-like regions, can be generated by a sequence of local even Majorana fermion operators such that for every state obtained in the sequence its energy is $O(1)$ above the ground-state energy. This argument shows that the energy barrier between logical states is $O(1)$, see [7].

Note that Lemma 3 can be applied to 1D geometry as well by considering a 2D lattice of size $L \times 1$. In this situation the strips A_i become intervals of a 1D chain of length $r-1$. Obviously, if a 1D code satisfies case (i) of Lemma 3, it does not provide any protection at all, since it has an even logical operator of constant weight and constant diameter. On the other hand, a 1D code satisfying case (ii) of Lemma 3 behaves similar to the Kitaev's 1D model, see Eq. (1). Indeed, such a code has two constant-weight odd logical operators \bar{C}_i, \bar{C}_j supported on some disjoint intervals A_i, A_j . Clearly, the largest possible diameter of the corresponding even logical operator $\bar{C}_i \bar{C}_j$ is of order L . Using the notation of Section 4, any 1D Majorana code must obey

$$d = O(1) \quad \text{and} \quad l_{\text{even}} = O(L). \quad (9)$$

In particular, it shows that Kitaev's 1D Majorana chain demonstrates the optimal behavior even among the subclass of *interacting* fermionic models corresponding to Majorana fermion codes.

In Section 7 we construct a 2D Majorana fermion code that demonstrates the optimal behavior allowed by Lemma 3. This code has a single even logical operator and two odd logical operators with weight of order L located on the opposite boundaries of the lattice.

Lemma 3 can be straightforwardly generalized to any spatial dimension D using the partition $\Lambda = A_1 \cup A_2 \cup \dots \cup A_t$ into disjoint hyper-strips of width at least r , that is, rectangles of size $s \times L \times \dots \times L$, where L is the lattice size and $s \geq r$.

In the rest of the section we prove Lemma 3.

Proof: Let us say that a subset of the lattice $M \subseteq \Lambda$ is *cleanable* iff for any logical operator $\bar{D} \in \mathcal{C}(\mathcal{S}_{\text{maj}}) \setminus \mathcal{S}_{\text{maj}}$ there exists a stabilizer $S \in \mathcal{S}_{\text{maj}}$ such that $\bar{D}S$ acts trivially on M , that is, $\text{Supp}(\bar{D}S) \cap M = \emptyset$. Otherwise we shall say that M is *uncleanable*. We shall use the following simple fact.

Proposition 2 *A subset $M \subseteq \Lambda$ is uncleanable iff there exists a logical operator $\bar{C} \in \mathcal{C}(\mathcal{S}_{\text{maj}}) \setminus \mathcal{S}_{\text{maj}}$ supported on M .*

The proposition can be easily proved by applying the Cleaning Lemma from [7] to the doubled stabilizer code constructed from \mathcal{S}_{maj} as described in Section 5. For the sake of completeness, we give a more direct proof of Proposition 2 in Appendix A.

Now consider two cases:

- (1) There are at least two uncleanable strips $A_i, A_j, i \neq j$;
- (2) There is at most one uncleanable strip A_i .

Consider first case (1). Let \bar{C}_i and \bar{C}_j be the logical operators of \mathcal{S}_{maj} supported on A_i and A_j which exist by Proposition 2. If at least one of \bar{C}_i, \bar{C}_j is even, we arrive at case (i) of the lemma. If both \bar{C}_i, \bar{C}_j are odd, we arrive at case (ii).

Let us now consider case (2). We shall color the strips in black and white in the alternating order such that the only uncleanable strip A_i (if any) is black. Then every white strip is cleanable. Moreover, since the generators of \mathcal{S}_{maj} have diameter smaller than the width of a strip, the union of all white strips is also cleanable. As was mentioned in Section 4, we can always choose at least one even logical operator $\bar{D} \in \mathcal{C}(\mathcal{S}_{\text{maj}}) \setminus \mathcal{S}_{\text{maj}}$. Let $S \in \mathcal{S}_{\text{maj}}$ be the stabilizer that cleans \bar{D} from the union of white strips. Then $\bar{C} := S\bar{D}$ is an even logical operator of \mathcal{S}_{maj} that has support only on black strips. Let \bar{C}_i be the restriction of \bar{C} onto a black strip A_i . Note that for any i the operator \bar{C}_i is either a stabilizer or a logical operator of \mathcal{S}_{maj} . If there exists a black strip A_i such that \bar{C}_i is a logical operator with even weight, we arrive at case (i) of the lemma. Otherwise, the number of black strips A_i such that \bar{C}_i is a logical operator with odd weight must be non-zero and even (recall that the overall weight of \bar{C} is even). Hence we can choose a pair of black strips A_i, A_j such that \bar{C}_i and \bar{C}_j are logical operators with odd weight. We arrive at case (ii) of the lemma. \square

7. Majorana color code

In this Section we describe a fermionic version of the topological color codes introduced by Bombin and Martin-Delgado in [31]. The color codes are weakly self-dual CSS codes with geometrically-local generators. One can define a color code on any two-dimensional lattice or, more generally, on any surface graph which is 3-valent and admits a 3-coloring of its faces. For such graphs two faces share an *even* number of vertices, as is easily checked. Given such a graph, the color code is defined by placing qubits at the vertices of the graph. The generators of the stabilizer group are associated with faces of the lattice. Specifically, for every face f one defines a pair of generators $S_f(X)$ and $S_f(Z)$ equal to

the product of Pauli X 's and Z 's respectively over all qubits lying on the boundary of f . The even overlap condition guarantees that all generators pairwise commute. One can show that the logical operators of the color code can be identified with homologically non-trivial loops on the lattice, see [31] for details.

Lemma 2 allows one to identify any color code with a doubled Majorana fermion code. A simple example is the 2D color code on a hexagonal lattice with periodic boundary conditions (i.e. a torus). Such code encodes 4 logical qubits, see [31]. The corresponding Majorana fermion code has a single Majorana mode c_u at every site of the lattice and a single generator C_f at every hexagon f . The generator C_f is proportional to the product of single-mode operators c_u over all sites u lying on the boundary of f . Note that the pair of generators $S_f(X), S_f(Z)$ can be obtained by applying the doubling transformation of Lemma 2 to the generator C_f . Hence the Majorana code with stabilizers C_f encodes 2 logical qubits. It is important to note that all 2D color codes discussed in Ref. [31] have only even-weight logical operators (with the exception of the so-called triangular codes which we discuss in Section 8). The absence of odd-weight logical operators in a color code implies the absence of odd logical operators in its fermionic version. Hence superselection rules do not play a role in their stability properties.

The formalism developed by Bombin and Martin-Delgado in [31] employs 3-coloring of the set of the faces of the lattice to classify the logical operators of the code. As we shall see, the global face 3-coloring condition is too restrictive as it leaves many interesting color-type codes beyond the scope of the formalism. In particular, the Majorana color codes that we describe below are defined on lattices that admit only a *local* 3-coloring, meaning that any topologically trivial region of the lattice admits a face 3-coloring but it cannot be extended to the entire lattice. The peculiar feature of such codes is that they possess odd logical operators.

Let $\Sigma = S^1 \times [0, 1]$ be a two-dimensional cylinder and $G \subseteq \Sigma$ be a graph embedded in Σ . We shall assume that the graph G induces a cellular decomposition of Σ , that is, the surface Σ can be decomposed into a set of faces, edges, and vertices that we shall denote F , E , and V respectively. The boundary $\partial\Sigma$ consists of two cycles $S^1 \times \{0\}$ and $S^1 \times \{1\}$. In order to define a Majorana color code, we shall impose four conditions on the graph G :

- (G1) The total number of vertices is even.
- (G2) Each vertex has degree 3 (trivalent graph).
- (G3) The boundary of any face has even length.
- (G4) The boundaries $S^1 \times \{0\}$ and $S^1 \times \{1\}$ have odd length.

Given a face $f \in F$, let $V(f) \subseteq V$ be the set of all vertices that lie on the boundary of f . We shall say that a face $f \in F$ is adjacent to a vertex $u \in V$ iff $u \in V(f)$, and we say that two faces are adjacent if they share a common edge. The assumption that G induces a cellular decomposition of Σ together with condition (G2) imply that any vertex $u \notin \partial\Sigma$ has exactly three adjacent faces and any vertex $u \in \partial\Sigma$ has exactly two

adjacent faces. This, together with (G4), implies via Proposition 1 that we must have odd logical operators. We show a non-trivial example of a surface graph G satisfying conditions (G1-G4) in Fig. 1.

Suppose each vertex $u \in V$ is occupied by a Majorana mode c_u . For any face $f \in F$ we define a *face operator*

$$C_f = \prod_{u \in V(f)} c_u. \quad (10)$$

Conditions (G2) and (G3) imply that all face operators have even weight and any pair of face operators commute with each other. Using the standard stabilizer formalism one can show that there exists a choice of phase factors $\eta_f \in \{\pm 1, \pm i\}$, $f \in F$, such that operators $\eta_f C_f$ generate an Abelian subgroup $\mathcal{S}_{\text{maj}}(G) \subseteq \text{Maj}(2n)$ not containing minus identity, i.e.

$$\mathcal{S}_{\text{maj}}(G) = \langle \eta_f C_f, \quad f \in F \rangle. \quad (11)$$

is a Majorana fermion code. This code will be referred to as a *Majorana color code* associated with G . One can also regard the codespace of $\mathcal{S}_{\text{maj}}(G)$ as the ground subspace of a fermionic local Hamiltonian

$$H = - \sum_{f \in F} \eta_f C_f \quad (12)$$

(Although the coefficients η_f do not affect any parameters of the code, one needs to have an explicit expression for η_f to define the Hamiltonian model Eq. (12). Let us mention that one can explicitly compute η_f using the geometrical structures known as *Kasteleyn orientations* of a surface [32, 33, 34]).

We begin by describing the logical operators of the code. Let γ_0 and γ_1 be the two boundary components of Σ , that is,

$$\gamma_0 = V \cap (S^1 \times \{0\}) \quad \text{and} \quad \gamma_1 = V \cap (S^1 \times \{1\}). \quad (13)$$

Define operators

$$\bar{C}_\alpha = \prod_{u \in \gamma_\alpha} c_u, \quad \alpha = 0, 1. \quad (14)$$

Note that for any face f , the set $V(f)$ has even overlap with γ_α since one can regard γ_α as a boundary of an external face obtained by patching up the hole in Σ . Therefore \bar{C}_α commutes with any face operator. On the other hand, condition (G4) implies that \bar{C}_α has odd weight, and thus $\bar{C}_\alpha \notin \mathcal{S}_{\text{maj}}(G)$. We conclude that \bar{C}_α are logical operators of the code. In addition, since $\bar{C}_0 \bar{C}_1 = -\bar{C}_1 \bar{C}_0$, these are two independent logical operators. In other words, the Majorana color code encodes at least one qubit and the logical Pauli operators for this qubit can be chosen as

$$\bar{X} = \bar{C}_0, \quad \bar{Y} = \bar{C}_1, \quad \bar{Z} = -i\bar{C}_0 \bar{C}_1. \quad (15)$$

The following lemma shows that this is the only logical qubit.

Lemma 4 *The Majorana color code has exactly one logical qubit, i.e., $k = 1$.*

Proof: Let us define a classical linear code $\mathcal{C} \subseteq \{0, 1\}^{|F|}$ whose codewords describe linear dependencies among the face operators. Specifically, a binary string $x = \{x_f\}_{f \in F}$ is a codeword of \mathcal{C} iff

$$\prod_{f \in F} C_f^{x_f} \propto I. \quad (16)$$

Using the standard stabilizer formalism one can show that

$$k = \frac{|V|}{2} - \dim(\mathcal{S}_{\text{maj}}(G)) = \frac{|V|}{2} - |F| + \dim(\mathcal{C}) = \dim(\mathcal{C}), \quad (17)$$

where the last equality follows from the Euler formula $|V| + |F| - |E| = 0$ and the identity $3|V| = 2|E|$. Since we have already shown that $k \geq 1$, we know that Eq. (16) has at least one non-trivial solution, that is,

$$\dim(\mathcal{C}) \geq 1. \quad (18)$$

Let u be a vertex in $\partial\Sigma$, and let f and g be the two adjacent faces. We claim that if we fix the values of x_f and x_g we uniquely determine a solution to Eq. (16). Since we see from Eq. (16) that $x_f \oplus x_g = 0$, we then proved that there are at most two solutions, and hence $\dim \mathcal{C} \leq 1$.

To prove the claim, let $v \in V$ be another vertex of G , and let $\omega = (u_0 = u, u_1, \dots, u_t = v)$ be any path connecting u and v . Suppose we have already set the value of x on some pair of faces f_i, g_i adjacent to the vertex u_i for some $i \geq 0$. Consider two cases. *Case 1:* $u_i \notin \partial\Sigma$. Let h_i be the third face adjacent to u_i . From Eq. (16) we infer that $x_{f_i} \oplus x_{g_i} \oplus x_{h_i} = 0$ which uniquely sets x_{h_i} . Since two of the faces f_i, g_i, h_i are adjacent to the edge (u_i, u_{i+1}) , it sets the value of x on some pair of faces adjacent to u_{i+1} . *Case 2:* $u_i \in \partial\Sigma$. In this case u_i has only two adjacent faces f_i, g_i . If $u_{i+1} \notin \partial\Sigma$, then both faces f_i, g_i are adjacent to u_{i+1} . If $u_{i+1} \in \partial\Sigma$ then only one of the faces f_i, g_i is adjacent to u_{i+1} , say the face f_i . Let $f_{i+1} = f_i$ and g_{i+1} be the two faces adjacent to u_{i+1} . From Eq. (16) we infer that $x_{f_{i+1}} \oplus x_{g_{i+1}} = 0$ which sets the value of x on the two faces adjacent to u_{i+1} . This actually even shows that x has to have the same value for all faces along a common boundary of Σ .

Applying this argument inductively one sets the value of x on some pair of faces adjacent to v which sets the value of x on all faces adjacent to v . Since any face is adjacent to some vertex, it shows that there is at most one way to extend x_f and x_g to a solution of Eq. (16). \square

The unique non-trivial solution of Eq. (16) constructed above allows one to define subsets of faces $F_0 = \{f \in F : x_f = 0\}$ and $F_1 = \{f \in F : x_f = 1\}$ such that any vertex has exactly two adjacent faces from F_1 , see Fig. 1 in which the faces from F_0 are represented by shaded hexagons. In other words, we have the following corollary.

Corollary 2 *There exists a unique partition of the set of faces F into disjoint subsets F_0 and F_1 such that each vertex has exactly two adjacent faces from F_1 and each vertex not lying on the boundary $\partial\Sigma$ has exactly one adjacent face from F_0 .*

Another interesting corollary of Lemma 4 is that the graph G is not face 3-colorable. Recall that a face 3-coloring is a mapping $c : F \rightarrow \{0, 1, -1\}$ such that for any pair of adjacent faces f, g one has $c(f) \neq c(g)$.

Corollary 3 *The graph G does not permit face 3-coloring.*

Proof: Indeed, suppose such a 3-coloring exists. Then clearly all faces in F_0 must have the same color, say, $c(f) = 0$ for all $f \in F_0$. It implies that all faces adjacent to the boundary $\partial\Sigma$ must be colored by ± 1 . However, since the boundary components have odd length, such a coloring does not exist. \square

One can use Corollary 2 to define a family of even-weight logical operators whose supports have geometry of a string connecting the two boundary components of Σ . Indeed, define a subset of edges

$$E_0 = \{e \in E : \text{ both faces adjacent to } e \text{ belong to } F_1\}. \quad (19)$$

Any edge $e \in E_0$ connects some pair of distinct faces in F_0 , or connects some face in F_0 with one of the two external faces $f_{\text{ext},0}, f_{\text{ext},1}$ obtained by patching the holes in Σ (see Fig. 1 where the edges from E_0 are shown in blue). Given any edge $(u, v) \in E_0$ connecting some pair of faces $f, g \in F_0$, the operator $c_u c_v$ commutes with any face operator C_f , $f \in F_1$, and anticommutes with C_f and C_g . Hence we can construct logical operators associated with paths of edges in E_0 that connect the two external faces $f_{\text{ext},0}$ and $f_{\text{ext},1}$. More specifically, consider a graph $G^{(0)}$ with a set of vertices $F_0 \cup f_{\text{ext},0} \cup f_{\text{ext},1}$ and a set of edges E_0 . Let $\gamma = (e_1, \dots, e_m)$, $e_i \in E_0$, be any path on $G^{(0)}$ connecting $f_{\text{ext},0}$ and $f_{\text{ext},1}$. Then the operator

$$\bar{C}_\gamma = \prod_{(u,v) \in \gamma} c_u c_v \quad (20)$$

commutes with all face operators C_f , $f \in F$. On the other hand, \bar{C}_γ anticommutes with \bar{c}_0 and \bar{c}_1 since it shares exactly one vertex with the boundaries γ_0 and γ_1 . We conclude that \bar{C}_γ is the logical operator $\bar{Z} \sim \bar{c}_0 \bar{c}_1$, see Eq. (15).

The graph $G^{(0)}$ defined above allows one to construct a face 3-coloring for any topologically trivial region of the lattice. Indeed, as was pointed out above, all faces $f \in F_0$ must have the same color, for instance, $c(f) = 0$ for all $f \in F_0$. Then one needs to color the faces $f \in F_1$ using the colors $c(f) = \pm 1$ such that adjacent faces in F_1 have different colors. Recall that any pair of adjacent faces in F_1 can be identified with some edge $e \in E_0$, see Eq. (19). Hence G admits a face 3-coloring iff the graph dual to $G^{(0)}$ admits a vertex 2-coloring. Let us denote this dual graph $G^{(1)}$. It has the set of vertices F_1 and the set of edges E_0 . The set of faces of $G^{(1)}$ can be identified with F_0 . Since each face $f \in F_0$ has even-length boundary, any homologically trivial cycle in $G^{(1)}$ must have even length. Hence one can construct a vertex 2-coloring of any subgraph $G^{(1)}$ that does not contain homologically non-trivial cycles.

Let us now bound the distance of the Majorana color code focusing on the physically relevant case when the generators of the code are *geometrically-local*. We shall assume that Σ is equipped with a metric such that the boundary components $S^1 \times \{0\}$ and

$S^1 \times \{1\}$ have length R , while the distance between them is L . We also assume that edges of G have length at most $O(1)$ and any face consists of $O(1)$ edges. Below we prove that the distance of the code grows linearly with the smallest of the surface dimensions, namely,

$$d = \Omega(\min(R, L)). \quad (21)$$

As for the minimum diameter of even logical operators, we prove the bound

$$l_{\text{even}} = \Omega(L), \quad (22)$$

see Section 4 for notations. The regime in which $R = O(1)$ while $L \gg 1$ can be regarded as protection by the superselection rules only since in this regime the code behave similarly to the Kitaev's 1D model, see Eq. (9). The regime in which both dimensions $R, L \gg 1$ are of the same order can be regarded as protection by the code distance only, since in this regime both even and odd logical operators are equally difficult to implement. In the intermediate regimes the code combines both types of protection in a way that can be controlled by the choice of R and L .

Let us now prove the bounds Eqs. (21,22). We start from observing that any odd logical operator must have weight $\Omega(R)$. Indeed, any such logical operator \bar{P} must anti-commute with even logical operators \bar{C}_γ constructed above, see Eq. (20). Obviously, one can choose m pairwise disjoint paths $\gamma_1, \dots, \gamma_m$ on the graph $G^{(0)}$ connecting the two external faces where $m = \Omega(R)$. Then the support of \bar{P} must have odd overlap with each of the paths γ_i , $i = 1, \dots, m$, that is, the weight of \bar{P} must be at least m .

Suppose now that \bar{P} is the minimum-weight operator among the even logical operators. Let $r = O(1)$ be the largest diameter of faces $f \in F$. Consider two cases: (i) the support of \bar{P} can be partitioned into two disjoint components separated by a distance greater than r ; (ii) such a partition does not exist. In the case (i) we have a decomposition $\bar{P} = \bar{P}_1 \bar{P}_2$, where \bar{P}_1, \bar{P}_2 individually commute with any face operator and at least one of \bar{P}_1, \bar{P}_2 is a non-trivial logical operator. Note that \bar{P}_1, \bar{P}_2 cannot be even operators since it would contradict the weight minimality of \bar{P} . Hence both \bar{P}_1 and \bar{P}_2 are non-trivial odd logical operators. However we have already shown that such operators must have weight $\Omega(R)$, that is, we arrive at $|\bar{P}| = |\bar{P}_1| + |\bar{P}_2| = \Omega(R)$. Let us now consider case (ii). Since \bar{P} must anti-commute with both \bar{c}_0, \bar{c}_1 , its support must have odd overlap with both γ_0 and γ_1 . Condition (ii) then implies that \bar{P} must have weight $\Omega(L/r) = \Omega(L)$. In both cases the diameter of the support of \bar{P} is $\Omega(L)$ since its overlaps with both γ_0 and γ_1 .

8. Other constructions of Majorana fermion codes with odd logical operators

In this section we discuss some alternative strategies to construct Majorana fermion codes with odd logical operators. Let us begin by considering an unphysical situation when the total number of Majorana fermion modes is *odd*. Then the operator $C_{\text{all}} = \prod_{u \in \Lambda} c_u$ is odd and therefore does not belong to \mathcal{S}_{maj} . On the other hand, C_{all} has

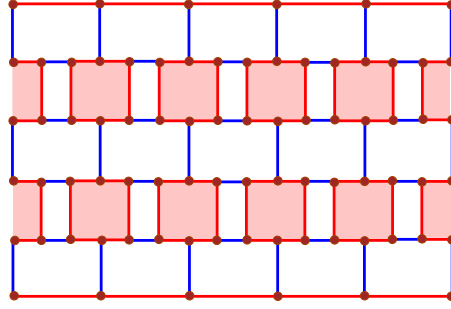


Figure 1. Example of a surface graph satisfying conditions (G1)-(G4). The lattice has periodic boundary conditions along the horizontal axis and open boundary conditions along the vertical axis. The subset F_0 consists of 10 faces (shaded hexagons). The edges of E_0 and E_1 are shown using the blue and red color respectively. The boundary components γ_0, γ_1 consist of 5 vertices.

even overlap with any stabilizer and hence $C_{\text{all}} \in \mathcal{C}(\mathcal{S}_{\text{maj}}) \setminus \mathcal{S}_{\text{maj}}$, that is, C_{all} is an odd logical operator. Note however that when the total number of Majorana fermion modes is odd, we encode *half-integer* number of qubits. For example, C_{all} can be the only operator in $\mathcal{C}(\mathcal{S}_{\text{maj}}) \setminus \mathcal{S}_{\text{maj}}$. Given the mapping between weakly self-dual CSS codes and Majorana fermion codes, see Section 5, it is then easy to construct such Majorana fermion codes encoding half-integer number of qubits. Indeed, let us take any weakly self-dual CSS code $[[n, k, d]]$ where the total number of qubits n is odd. Viewed as a Majorana fermion code (see Lemma 2) it encodes $k/2$ logical qubits and has n Majorana modes. For example, we could take Steane's $[[7, 1, 3]]$ code [22] encoding a single qubit. The corresponding Majorana fermion code has three generators, $\mathcal{S}_{\text{maj}} = \langle S_1, S_2, S_3 \rangle$, where $S_1 = c_1 c_3 c_5 c_7$, $S_2 = c_2 c_3 c_6 c_7$, and $S_3 = c_4 c_5 c_6 c_7$. The logical \bar{X} and \bar{Z} operator for the Steane code become a single logical operator \bar{C}_{all} for the Majorana fermion code which encodes half a qubit. Now we can take another copy of this code, or another weakly self-dual CSS code with an odd number of qubits and take the product of these codes $\mathcal{S}_{\text{maj}}^1 \times \mathcal{S}_{\text{maj}}^2$. We now have an even number of Majorana fermion modes, hence an integer number of encoded qubits. At the same time, odd logical operators of $\mathcal{S}_{\text{maj}}^1$ and $\mathcal{S}_{\text{maj}}^2$ give rise to odd logical operators of the product code $\mathcal{S}_{\text{maj}}^1 \times \mathcal{S}_{\text{maj}}^2$. For two copies of the Steane code, the logical \bar{X} is a weight-3 Majorana fermion operator on the 7 modes of the first Steane code and the \bar{Z} is the same operator on the 7 modes of the second Steane code. This plug-and-play procedure of adding halves of qubits living on separate spatial supports can be enhanced by inserting a piece of passive material which encodes no qubits between the two coding regions. The (linear) size of this passive region determines the minimum diameter of even logical operators l_{even} . In this way Kitaev's 1D Majorana fermion model can be viewed as a combination of the trivial code comprising of a single mode labelled '1', a piece of passive material including modes 2 to $2L - 1$ in which the Majorana fermions are paired, and again a trivial code on mode $2L$. Such procedure could for example also be applied to another class of 2D color codes, namely the triangular codes [31] which encode a single qubit and hence half a qubit

when the code is viewed as a Majorana fermion code.

Acknowledgements

BMT and SB acknowledge support by the DARPA QUEST program under contract number HR0011-09-C-0047. BL acknowledges the financial support and the warm hospitality from IBM Research and its employees.

Appendix A

In this Appendix we prove Proposition 2.

Proof: Given a subset of modes $M \subseteq \Lambda$, we shall define two subgroups of \mathcal{S}_{maj} . The first subgroup denoted as $\mathcal{S}_{\text{maj}}(M)$ contains all elements of \mathcal{S}_{maj} whose support is contained in M . The second subgroup denoted as $\mathcal{S}_{\text{maj}}^M$ contains all operators $P \in \text{Maj}(2n)$ whose support is contained in M that can be *extended* to some stabilizer. In other words, $P \in \mathcal{S}_{\text{maj}}^M$ iff $\text{Supp}(P) \subseteq M$ and $PR \in \mathcal{S}_{\text{maj}}$ for some operator $R \in \text{Maj}(2n)$ such that $\text{Supp}(R) \cap M = \emptyset$. By definition, one has $\mathcal{S}_{\text{maj}}(M) \subseteq \mathcal{S}_{\text{maj}}^M$.

We shall use the parameterization $\phi : \text{Maj}(2n) \rightarrow \{0, 1\}^{2n}$ constructed in Section 5, see Lemma 2. Consider the linear subspaces (classical codes)

$$C = \phi(\mathcal{S}_{\text{maj}}), \quad C(M) = \phi(\mathcal{S}_{\text{maj}}(M)), \quad \text{and} \quad C^M = \phi(\mathcal{S}_{\text{maj}}^M). \quad (23)$$

By definition, one has the inclusion $C(M) \subseteq C^M$. Since the code C is weakly self-dual, that is, $C \subseteq C^\perp$, one has $\sum_{u \in \Lambda} x_u y_u = 0$ for all $x \in C(M)$ and $y \in C$. However, since x has support only on M , it translates into $\sum_{u \in M} x_u y_u = 0$, that is, we have also an inclusion

$$C(M) \subseteq (C^M)^\perp. \quad (24)$$

By abuse of notations, from now on we shall consider the codes $C(M)$ and C^M as linear subspaces of $\{0, 1\}^m$, where $m = |M|$ (note that any vector in $C(M)$ or C^M has all zeros outside of M).

There are two possibilities. First, the inclusion Eq. (24) is an equality, that is, $C(M) = (C^M)^\perp$. Taking the orthogonal complement of both sides we get

$$C^M = C(M)^\perp. \quad (25)$$

Let $\bar{C} \in \mathcal{C}(\mathcal{S}_{\text{maj}}) \setminus \mathcal{S}_{\text{maj}}$ be any logical operator and $x = \phi(\bar{C})$. Decompose x as $x = x_{\text{int}} \oplus x_{\text{ext}}$, where x_{int} and x_{ext} have support inside and outside M respectively. Since \bar{C} commutes with any stabilizer supported on M we conclude that $x_{\text{int}} \in C(M)^\perp$ and hence Eq. (25) implies $x_{\text{int}} \in C^M$. It means that $\phi^{-1}(x_{\text{int}})$ can be extended to some stabilizer $S \in \mathcal{S}_{\text{maj}}$. Then $\bar{C}S$ acts trivially on M . Hence M is cleanable.

The second possibility is that the inclusion Eq. (24) is strict. Then there exists some $x \in (C^M)^\perp$ such that $x \notin C(M)$. Let $\bar{C} = \phi^{-1}(x)$. Then \bar{C} has support on M , commutes with any element of \mathcal{S}_{maj} , but does not belong to \mathcal{S}_{maj} . Hence \bar{C} is a logical operator supported on M .

To summarize, we have shown that if M is uncleanable then the inclusion Eq. (24) must be strict and hence there must exist a logical operator supported on M .

Let us now prove the converse. Suppose \bar{C} is a logical operator supported on M . If \bar{C} is odd, then M is uncleanable. Indeed, any stabilizer $S \in \mathcal{S}_{\text{maj}}$ must have even overlap with $\text{Supp}(\bar{C})$ and thus the support of $\bar{C}S$ contains odd number of modes (and hence at least one) from $\text{Supp}(\bar{C}) \subseteq M$. If \bar{C} is even, then there must exist a logical operator \bar{C}' that anti-commutes with \bar{C} . Let us show that $\bar{C}'S$ acts non-trivially on $\text{Supp}(\bar{C}) \subseteq M$ for any stabilizer $S \in \mathcal{S}_{\text{maj}}$ which would imply that M is uncleanable. Indeed, since \bar{C} is even and anti-commutes with $\bar{C}'S$, the overlap between $\text{Supp}(\bar{C})$ and $\text{Supp}(\bar{C}'S)$ must be odd and hence $\text{Supp}(\bar{C}'S)$ contains at least one mode from $\text{Supp}(\bar{C}) \subseteq M$. Thus in both cases M is uncleanable. \square

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